Time–Frequency Analysis of Variable Star Light Curves

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(**Eleventh Annual Florida Workshop in Nonlinear Astronomy and Physics**
NONLINEAR SIGNAL AND IMAGE ANALYSIS
(30 November - 2 December 1995.))

I. INTRODUCTION

The determination of the period variations of stellar light curves is a classical problem in astronomy. The so-called O–C method has been widely used to search for any deviation from the strictly repetitive occurrence of the maxima or minima of the light curves. The plot of the observed (O) minus the calculated (C) time of the extrema – the O–C diagram provides useful information about long term period changes and even about the motion of the variable star in a binary system. This method, however, breaks down when the signal contains more than one frequency or it has strong modulations.

The different forms of frequency spectra (mainly the Fourier–transform) became an important tool for the analysis of multiperiodic variations and the Fourier decomposition of the periodic light curves of Cepheids and RR Lyrae stars turned into a powerful device for the comparison of observations and theoretical models. The astronomical data are usually gapped and unevenly sampled because of the rhythm of days and nights, the weather conditions and telescope time availability. These sampling properties introduce lots of complications for the period determination, like the aliasing due to the spectrum of the observing window. This is one of the reasons that the comparison of the Fourier spectra of different observational segments has been the principal method for investigating the change in the periods and amplitudes of oscillations for a long time.

The world wide observational campaigns for short period stars (like white dwarf and δ Scuti stars) and the collection of mostly amateur observations of long period (RV Tauri, Mira, Semiregular) variables have made it possible to use more sophisticated tools of time-frequency analysis. In the last years the wavelet analysis with the Morlet kernel [1] has been applied to light variations. In the following we take into consideration the wavelet only as a tool for time–frequency description. For the application in scaling and finding self similar behavior we refer to [2] and Scargle in these proceedings.

The Morlet wavelet was first introduced in the astronomical literature by Goupil *et al.* [3], where they applied it to the light curves of white dwarf stars. Later the method was tested for synthetic signals representing typical astronomical observations by Szatmáry *et al.* [4]. Semiregular, Mira and W Virginis type variables were recently investigated with the method [5–8]. The wavelet analysis was also performed on the solar p-modes [9] and the variation of the solar cycle [10]. A similar method, the Gábor–transform was used to analyze frequency variation of an Ap star [11].

While the Morlet wavelet has been the primary tool in variable star astronomy, other time-frequency descriptions, like the generalized Wigner distribution (see *e.g.* [12,13]) have been successfully applied for engineering problems. The main purpose of this paper is to compare the performance of these different methods on variable star data. In Section 1. we give an introduction to the Morlet wavelet and Gábor transforms. The time frequency distributions are shortly described in Section 2. and we present the analysis of observational data in Section 3.

II. THE GÁBOR AND THE WAVELET TRANSFORMS

The classical tool of time-frequency analysis is the short-time Fourier transform (STFT), frequently referred to also as windowed Fourier transform. The signal $f(t)$ is weighted by a time-localized window function $h(t)$ and then Fourier transformed:

$$F(t,\nu) = \int_{-\infty}^{+\infty} f(\tau)h^*(\tau - t)\exp(-2i\pi\tau\nu)d\tau. \tag{1}$$

The STFT was introduced by Gábor [14] with the Gaussian analyzing window:

$$h(t) = \exp(-t^2/(2\sigma^2)). \tag{2}$$

We refer to this time–frequency representation as the *Gábor transform* (GT). (The general form of the STFT is also frequently called GT, independently of the choice of the window.) The spectrogram is the power spectrum version ($|F(t,\nu)|^2$) of the STFT [12].

The *wavelet transform* (WT) of a time series is defined as:

$$S(t,a) = a^{-1/2} \int_{-\infty}^{+\infty} f(\tau) g^*(\frac{\tau - t}{a}) dt, \tag{3}$$

where $g(t)$ is the kernel of the wavelet transformation. The variable a corresponds to the scale (period) parameter. We use the following simplified Morlet wavelet kernel:

$$g(x) = e^{-x^2/2+icx}, \tag{4}$$

where the parameter c controls the ratio of the time and frequency resolution. (Its usual value is 2π.) As in our previous works [8,15] we use a modified version of the wavelet transform, transforming it to a time–frequency representation instead of the time–scale version:

$$T(t,\nu) = a^{-1/2} S(t, a(\nu)), \tag{5}$$

where ν is the frequency. The $a(\nu) = c/(2\pi\nu)$ scaling gives the correct frequency of a periodic function at the maximum of the wavelet modulus *i.e.* the absolute value of the $T(t,\nu)$ and this maximum value is proportional to the signal amplitude. Thus for $x(t) = \cos(\omega t + \phi)$ one can obtain $|T| = (\pi/2)^{1/2} e^{-1/2(a\omega - c)^2}$.

It is more practical to calculate the wavelet from the Fourier transform ($F(\theta)$) of the signal. With the Morlet kernel this is given by

$$T^+(t,\nu) = \frac{1}{\pi} \int_0^{+\infty} F(\theta) \exp\left(-\frac{1}{2}\frac{(\theta-\nu)^2}{\sigma^2(\nu)}\right) \exp(2i\pi\theta t) d\theta, \tag{6}$$

where $\sigma(\nu) = \nu/c$. Here we have introduced T^+ calculating the one sided inverse Fourier-transform. (We discuss later why it is advantageous.) From this equation it is clear that the Morlet wavelet is given by a band pass filtering, with a Gaussian frequency transfer function centered at the analyzing frequency ν and with a ν dependent bandwidth. Since the value of the Gaussian at θ=0 is $\exp(-c^2/2)$, the overlap of the filter to the negative frequencies is negligible when c is grater than ≈ 3. In this case $T^+(t,\nu) \approx T(t,\nu)$. On the other hand, when c is very small, *i.e.* when the bandwidth of the filtering is larger than the bandwidth of the signal, T^+ reduces to the *analytic signal* of the function $f(t)$. (The analytic signal is a complex function whose real part is the signal and its Fourier transform vanishes for the negative frequencies. For the properties of the analytic signal see *e.g.* [12].) According to our definition, the negative frequency part of the Fourier transform of $T^+(t,\nu)$ vanishes for any fixed value of ν, *i.e.* T^+ is the analytic signal of the band pass filtered function.

The previously described properties of T^+ remain the same, even when we replace σ by a constant in Equation 6. Then the bandwidth of the filter is constant *i.e.* we get a filtered Fourier transform, which is equivalent to the Gábor transform apart from a factor of $\exp(2i\pi\nu t)$ (which disappears when one plots the absolute value or the power). What is then the difference between the two forms? The wavelet locates the high frequency components more precisely in time and is advantageous for example if one searches for sudden changes in the signal. On the other hand, the price we have to pay for the higher temporal resolution is a higher contamination in the higher frequency part of the spectrum (for higher frequencies the frequency–transfer function is wider). The WT is disadvantageous when the harmonics of the fundamental frequency (or the overtones of that mode) have much smaller amplitudes than the fundamental one. The situation is reversed for the GT.

For the numerical realization of both the GT and WT we used Equation 6. A fast inverse Fourier transform is calculated for all values of ν . For the GT we replace $\sigma(\nu)$ by $\sigma(\nu_0)$, where ν_0 is a constant frequency. By this definition the Gábor transform is matched to the WT at the frequency ν_0, *i.e.* the resolution is the same in both time-frequency maps at ν_0.

In most applications that have been performed on variable star data, generally only the modulus of the wavelet or Gábor transform has been investigated. However the phase of the transform gives important information about the frequency evolution of the signal, and it can provide a better estimation of the local frequency. The instantaneous frequency is defined by the time derivative of the phase of the analytic signal. Similarly one can define the instantaneous frequency around an average frequency ν_0 by:

$$\nu_{inst} = \frac{\partial}{\partial b}\varphi(\nu_0, b), \tag{7}$$

where $\varphi(\nu, b) = \arg(T^{+}(\nu, b))$. Here the frequency window is expected to be real. For a more rigorous derivation of the instantaneous frequency of STFT see [16].

III. TIME–FREQUENCY DISTRIBUTIONS

The generalized form of time frequency distributions (GTFD) was introduced by Cohen [13]:

$$C(t, \nu) = \frac{1}{2\pi}\int\int\int \exp(-i\theta t - 2i\pi\tau\nu + i\theta u)\Phi(\theta, \tau) f^{*}(u - \frac{\tau}{2}) f(u + \frac{\tau}{2}) du d\tau d\theta, \tag{8}$$

where $f(t)$ is the (generally complex) signal and $\Phi(\theta, \tau)$ is the kernel of the distribution. With the simplest kernel ($\Phi(\theta, \tau) = 1$) the definition of the GTFD reduces to the Wigner–distribution (WD):

$$W(t, \nu) = \int \exp(-2i\pi\tau\nu) f^{*}(t - \frac{\tau}{2}) f(t + \frac{\tau}{2}) d\tau, \tag{9}$$

The WD of the function $\exp(i\Omega t)$ is $W(t, \nu) = \delta(2\pi\nu - \Omega)$, giving a peak at the exact frequency only. However, since the transformation performs a nonlinear operation on the signal, multi–component (or multi periodic) signals have cross terms in the Wigner distribution. Similarly while the WD of an impulse ($\delta(t - t_0)$) is well localized at $t = t_0$, the combination of time localized functions can give nonzero WD even for times when the signal is vanishes. These properties of the Wigner distribution can be very disturbing when one investigates strongly modulated multiperiodic time series. To avoid this problem one can insert a kernel with localizing properties into the generalized time–frequency distribution.

For our applications we use the kernel defined by Choi and Williams [17] *i.e.* $\Phi(\theta, \tau) = \exp(-\theta^2\tau^2/\sigma)$, giving the following distribution

$$C(t, \nu) = \frac{1}{2\pi^{1/2}}\int\int (\tau^2\sigma)^{-1/2} \exp(-\sigma(u - t)^2/\tau^2 - 2i\pi\tau\nu) f^{*}(u - \frac{\tau}{2}) f(u + \frac{\tau}{2}) du d\tau. \tag{10}$$

Our numerical implementation is based on the discretized and windowed version of the Choi–Williams distribution (see [17], equation 20). It introduces two more parameters, the lengths of the window in the time and frequency domain (M and N respectively). In our calculations N is given by the maximum frequency, *i.e.* we fix this parameter. The length of the temporal window is given by M times the sampling time.

There is a straightforward connection between the time–frequency distribution and the spectrogram (Gábor transform). If one replaces the kernel $\Phi(\theta, \tau)$ by the *ambiguity function* (the two dimensional Fourier transform of the Wigner distribution) of the *window* of the spectrogram, then the resulting time–frequency distribution is the spectrogram itself [12]. Since the ambiguity function of a Gaussian is a two dimensional Gaussian, the Gábor transform is given by a time-frequency distribution with the kernel $\Phi(\theta, \tau) = c\exp(-\tau^2/(4\sigma^2) - \sigma^2\theta^2)$. From this it is clear that both the CWD and the spectrogram can be generated from the same time–frequency distribution by slightly different exponential kernels. Similarly, generalized time-scale energy distributions were introduced in [18] extending the wavelet-transform.

IV. APPLICATION TO VARIABLE STARS

In this section we present the time–frequency analyses of two variable star light curves. The raw data consist of the visual estimates of the brightness. To reduce the observational noise and produce an evenly sampled time sequence, we have first averaged the data in several-day-long bins, then interpolated by a smoothing spline (for this procedure see [15]). The comparison of the Fourier transforms of the smoothed and unsmoothed data (together with the corresponding spectral windows) indicates that this preprocessing does not alter the signal at the frequencies of interest. The sampling time of the smoothed data was 2 days for R Sct and 5 days for T Umi.

For all data sets we plot the square root of the positive part of the Choi Williams distribution. In this way we get the same amplitude scale as for the wavelet and Gábor transforms. (With a power scale the lower amplitude oscillations are hardly visible.)

A. T Ursae Minoris

T Ursae Minoris is a Mira star with a period of $\approx$ 300 days. The rapid decrease of the period of T Umi was reported by Gál and Szatmáry [6]. The period dropped from 314.5 days to 283.2 days during a 25 years period *i.e.* $\approx$ 30 pulsational cycles. Their wavelet plot clearly shows the variation of the frequency, and the investigation of the O–C diagram confirms this finding. Here we give a comparison of the different methods on this data set.

The wavelet and the Gábor transforms of the light variation are presented on the top of Figure 1. While the two maps are the same around the average frequency of the variation ($\approx$ 0.003 c/d) – guaranteed by the match of the window functions in that frequency, the Gábor transform resolves definitely better the harmonics of the pulsational frequency *i.e.* the higher temporal resolution of the the WT at higher frequencies is rather disadvantageous in this application. It is important, because the absolute change of the frequency is twice and three times larger at these harmonics. The Choi–Williams distribution (lower left box on Fig. 1.) emphasizes even better the second and third harmonics. The parameters of the CWD are σ=10 and M=128. For comparison we present the instantaneous frequency (evaluated from the Gábor transform with Eq. 7) in Figure 1. too. The solid line represents the frequency variation of the lowest frequency (f_0), while the dotted line marks one half of the instantaneous frequency at $2f_0$. The second curve is noisier, due to the smaller amplitude, but the increasing trend in the frequency manifests itself in the same way. It has not been possible to calculate the same for the third frequency.

To be honest we have to note that the O–C diagram really provides the same information on the basic period variation. The additional feature of the time–frequency maps is to exhibit the temporal variations of the different amplitudes. For example after t = 42,000 the amplitude at $f \approx 0.003$ c/d drops while the amplitude of the harmonics increases for the same short period of time. This event correlates with the beginning of the period change. This can be a fingerprint of the physical process modifying the nature of the star (see [6] for the explanation as a possible helium shell flash).

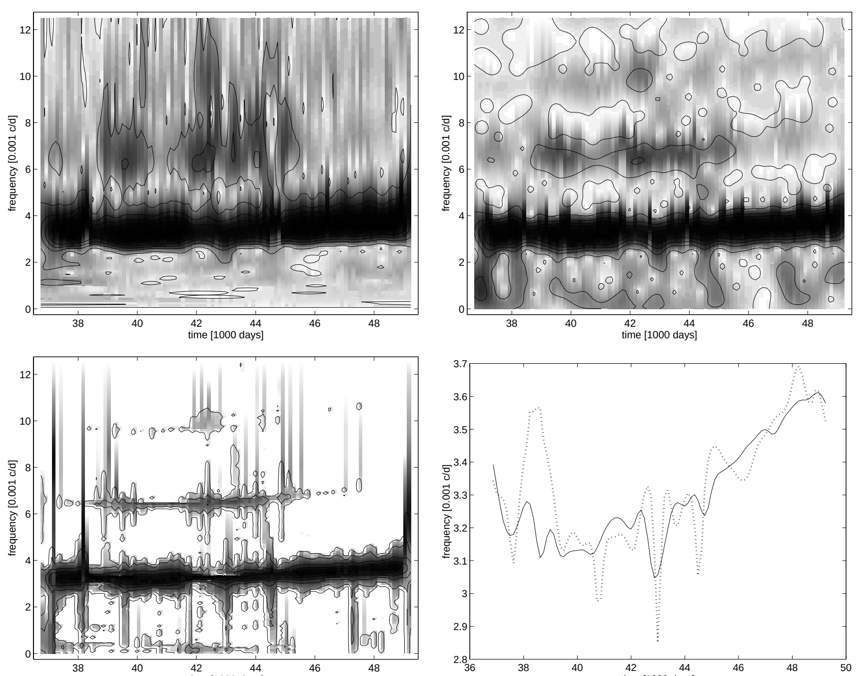

FIG. 1. *The wavelet map (upper left), Gábor transform (upper right), Choi–Williams distribution (lower left) and the instantaneous frequencies (lower right) of the light curve of T Umi*

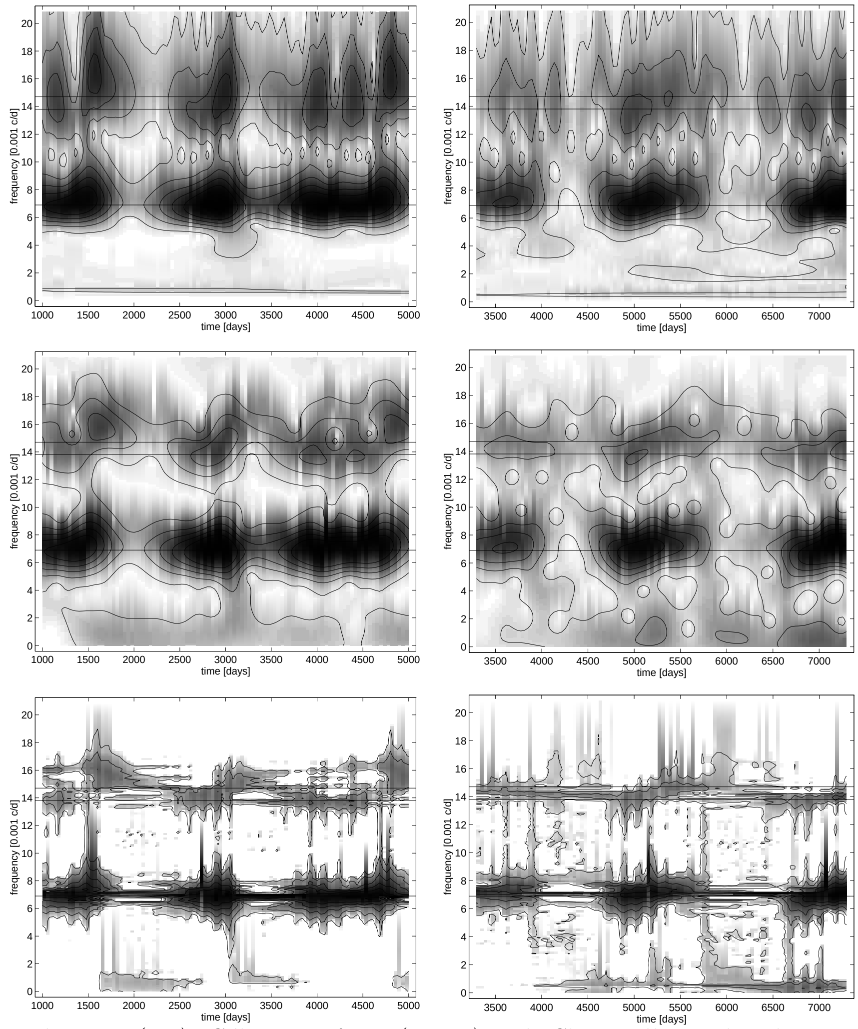

FIG. 2. *The wavelet map (top), Gábor transform (center) and, Choi–Williams distribution (bottom) of the synthetic data (left column) and the light curve of R Sct (right column)*

B. R Scuti

R Scuti is an RV Tauri type variable star, with irregular light variations. An application of the global polynomial flow reconstruction method to the light curve has found that the brightness variations can be modelled by a four dimensional flow or map [15] (see also Gousbet *et al.* in these proceedings and [19]).

The Fourier spectrum of the light curve displays two broad peaks near $\nu_1 \approx 0.069$ c/d and $\nu_2 \approx 0.0147$ c/d. These same frequencies show up in the linear stability analysis of the fixed point of this map which has two spiral manifolds, one unstable and the other stable. In the 4D phase-space of the system one first sees a spiralling out of the trajectory along the ν_1 unstable directions. During this part of the pulsations one expects a strong harmonic content at $2\nu_1$ because nonlinear effects distort the pulsations away from sinusoidal. This phase is ended when nonlinear effects cause a reinjection into the ν_2 stable manifold, giving rise to oscillations with a decaying amplitude, and now with frequency ν_2. One can thus expect a switch back and forth between the neighboring frequencies $2\nu_1$ and ν_2, and this is clearly seen in the synthetic signals (i.e. in the signals produced by an iteration of the map). These features were clearly visible in the wavelet analysis of the *synthetic signal* [15]. However, the same analysis for the real observations of R Sct by the WT was not informative at all. This is not astonishing because the light curve is contaminated by noise, and we have only a relatively short sample of the amplitude modulated signal, while from the clean synthetic signal one can chose an optimal segment. One of the motivations for this paper has precisely been a search for more sophisticated methods (or the combination of them) that enable one to display this behavior in the observational data.

Here we present a comparison of the analyses of the synthetic curve and of the R Scuti data by the three methods.

We have calculated all the transforms for a grid of 128 frequency and 100 time values. The value of $c = 2\pi$ has been used in the wavelet analysis and we have set σ in the Gábor transform to match the resolution of the wavelet at ν=0.007 c/d. The parameters of the Choi–Williams distribution are: σ=2 and M=128. The gray scale plots of the distributions are presented in Figure 2. The values of ν_1, $2\nu_1$ and ν_2 are indicated by horizontal lines in all figures. We confirm that it is hard to collect any information other than the amplitude modulation of the signal from the plots of the wavelet modulus. Only the plot of the instantaneous frequency sheds some light on the frequency variation of the synthetic signal (see [15]). On the other hand, the Gábor transform clearly shows the frequency shift for the synthetic signal, and there is some indication of it for the R Sct data, as well. The Choi–Williams distribution in contrast seems to be the superior for both data sets. The frequency changes between $2\nu_1$ and ν_2 are clearly visualized in the plots.

It is very pleasing to us that the application of this type of time-frequency analysis (with the Choi-Willliams distribution) provides a further test for a positive comparison of the synthetic data and the observed light curve of R Sct.

V. CONCLUSION

Astronomers have essentially limited themselves to the use of the wavelet transform in their investigations of the time–frequency characteristics of variable light curves. Here we have compared the results that one obtains with the wavelet transform to those of other time-frequency methods, using both real light curves and a synthetic signal. From these tests we can conclude (a) that the Gábor transform provides much more informative results on the high frequency part of these data than the wavelet transform, but (b) that the time-frequency analysis with the Choi–Williams distribution is definitely superior to both methods, at least on these data.

We have also shown that the time–frequency analysis can provide us with an important tool for the comparison of chaotic data sets. Thus, in the case of the irregular star R Scuti our global flow reconstruction found a wavering between two neigboring frequencies. The time-frequency analysis was able to show that this subtle effect is actually also present in the noisy observational light curve data, thus further strenghtening our conclusion that a low dimensional flow governs the dynamics of this star.

ACKNOWLEDGMENTS

We wish to thank Leon Cohen for helpful discussions, Károly Szatmáry for the light curve of T Umi and Janet Mattei for the AAVSO observational data. We gratefully acknowledge the support of NSF (grants AST92-18068 and INT94–15868).

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